THE MONTESSORI EVENT March 16-19, 2023 Boston & Online

Teaching Addition and Subtraction Facts to Diverse Learners

Joan A. Cotter, Ph.D.; Curriculum Developer; AMI

Cheryl Heatwole Shenk, M.Ed; Program Developer; AMS

Changes in Math in Last Century

- Expansion of mathematics itself
- Math becoming more visual
- Explosion of applications for math
- Development of tools for calculations
- Research into how our brains work

2

Rationale

"This book of methods compiled by one person alone, must be followed by many others. It is my hope that, . . . , other educators will set forth the results of their experiments. These are the pedagogical books which await us in the future."

– Maria Montessori

The Montessori Method: Scientific Pedagogy as Applied to Child Education in "The Children's Houses," 1912.

3

Visualizing

"Think in pictures, because the brain remembers images better than it does anything else."

–Ben Pridmore
Memory Champion, 2009

Dictionary Definition:
   The formation of a mental image of something.

A visualized image is similar to but more abstract than a physical object. It is not an exact replica.
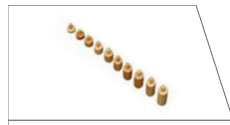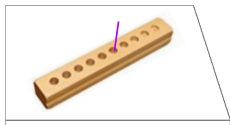
4

Visualizing
Japanese criteria for manipulatives

- Representative of structure of numbers.
- Easily manipulated by children.
- Imaginable mentally.

– Japanese Council of
Mathematics Education

5

Visualizing
Montessori example

6

Visualizing

Try to visualize 8 identical apples without grouping.

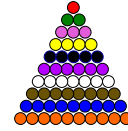Now try to visualize 8 apples: 5 red and 3 green.

7

Grouping in Fives
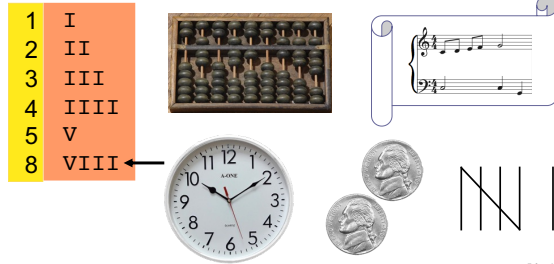
Subitizing
Colored bead bars

- Associating quantities with colors is not subitizing.
- Using colors involves different mental operations.

Grouping in Fives
Black and White Bead Stairs

"Grouped in fives so the child
does not need to count."

– A. M. Joosten
Director of Indian Montessori and Montessori Training Center of Minnesota

---

Grouping in Fives
Chunking

- Grouping in fives provides "chunking."
- According to psychologists, a chunked item can be stored or processed as one item.
- Ten can be reduced to two chunks.

Grouping in Fives
In Japan

- Students consistently group in fives.
- Children are discouraged from using counting for adding.
- They are not taught to count on.

Visualizing
Karen Wynn's research

- Infants can determine quantities long before they can count, or even talk.
- They can add and subtract 1 to 3 objects without counting.
- This ability to recognize quantities without counting is called *subitizing*.
- Subitizing quantities makes them *visualizable*.

Subitizing

- Five-month-old infants can subitize to 3.
- Three-year-olds can subitize to 5.
- Four-year-olds can subitize to 10.

14

Subitizing

- Subitizing "allows the child to grasp the whole and the elements at the same time." —Benoit
- Subitizing seems to be a necessary skill for understanding what the counting process means. —Glasersfeld
- Children who can subitize perform better in mathematics long term. —Butterworth
- Counting-on is a difficult skill for many children. —*Journal for Research in Math Educ.* Nov 2011
- Math anxiety affects counting ability, but not subitizing ability.

15

Adding by Counting
From a child's perspective

Because we're so familiar with 1, 2, 3, we'll use letters.

A = 1
B = 2
C = 3
D = 4
E = 5, and so forth

16

Adding by Counting
From a child's perspective

F + E =

17

Adding by Counting
From a child's perspective

F + E =

A B C D E F

18

Adding by Counting
From a child's perspective

F + E =

A B C D E F   A B C D E

19

Adding by Counting
From a child's perspective

$$F + E = K$$

A B C D E F   G H I J K

20

Adding by Counting
From a child's perspective

$$E + D =$$

Find the sum without counters or fingers.

21

Adding by Counting
From a child's perspective

Now memorize the facts!!
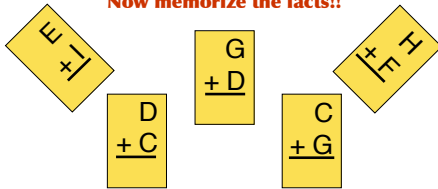
22

The Counting Model
Counting on

- Children are often expected to know what number comes next without starting from 1.
- Think of the nursery rhyme "Jack and Jill."
- Without starting from the beginning, what word comes after *hill*?

23

The Counting Model
Overview

- Is not natural; it takes years of practice.
- Provides poor concept of quantity.
- Ignores place value.
- Is very error prone.
- Is tedious and time-consuming.
- Does not provide efficient ways to master the facts.
- Is not predictive of future achievement in math.

24

The Counting Model
Difficulties in Counting

- Children with dyslexia or dyscalculia
- Children with SLI (specific language impairment)
- Children with memory challenges
- Children with poor motor control
- Children from low SES backgrounds

25

The Counting Model
Is Counting the Core of Math?

- Counting is used primarily in arithmetic.
- Arithmetic is one of about 200 branches.
- Abacuses made counting unnecessary.
- It is very slow for multiplying.
- Counting doesn't work for money or fractions.

26

The Counting Model
Counting is pervasive in Montessori materials.

- Number Rods
- Spindle Boxes
- Golden Bead materials
- Snake Game

- Dot Game
- Stamp Game
- Multiplication Board
- Bead Frame

27

Learning 1 to 10
Using fingers

Teach number names as a three-period lesson.

28

Learning 1 to 10
Yellow is the Sun

Photo by Cheryl Heatwole Shenk

29

Learning 1 to 10
Recognizing 5
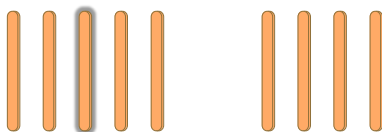
5 has a middle; 4 does not.

30

Learning 1 to 10

Photo by Cheryl Heatwole Shenk

31

Modifications
Number Rods
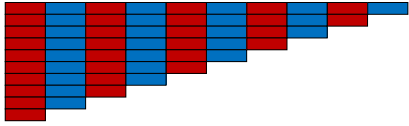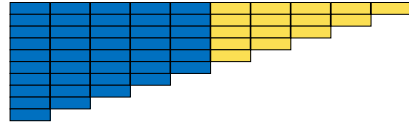
32

Modifications
Number Rods

33

Modifications
Number Rods

34

Modifications
Number Rods

35

Modifications
Spindle Box

35 blue spindles and 10 yellow spindles.

36
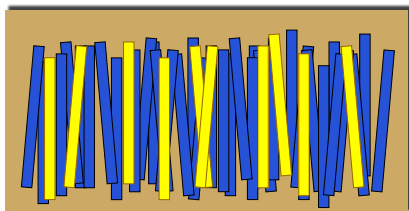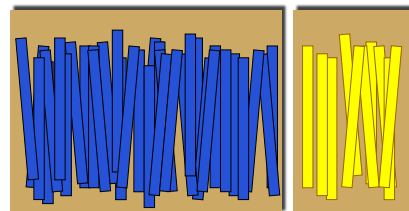
Modifications
Spindle Box

35 blue spindles and 10 yellow spindles.

37
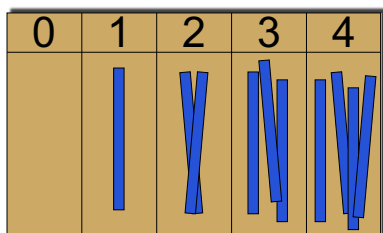
Modifications
Spindle Box

38
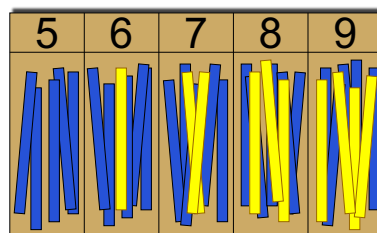
Modifications
Spindle Box

39

Modifications
Spindle Box

Photo by Cheryl Heatwole Shenk

40

Modifications
Spindle Box

Photos by Cheryl Heatwole Shenk

41

Modifications
Golden Bead Material
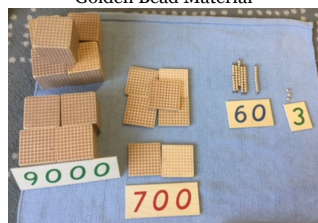
Photo by Cheryl Heatwole Shenk

42

Modifications
Stamp Game

43

Modifications
Stamp Game

44

Modifications
Stamp Game

45

Bead Frame

46

Modified Bead Frame

47

Learning 1 to 10
Entering quantities

48
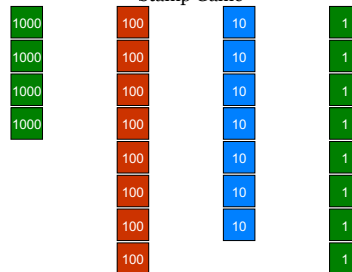
Learning 1 to 10
Entering quantities
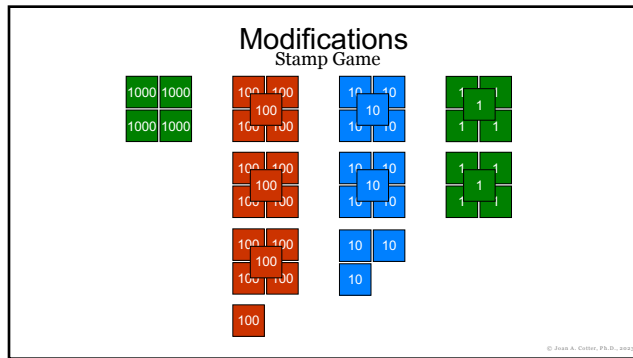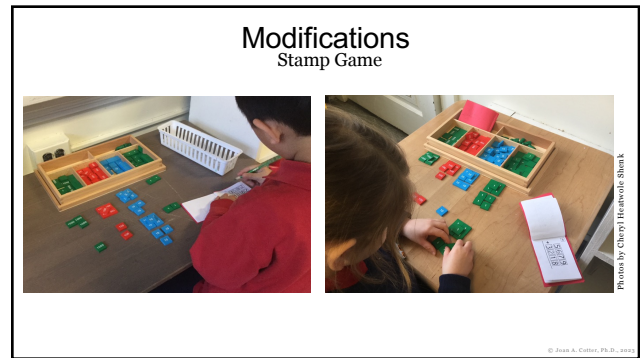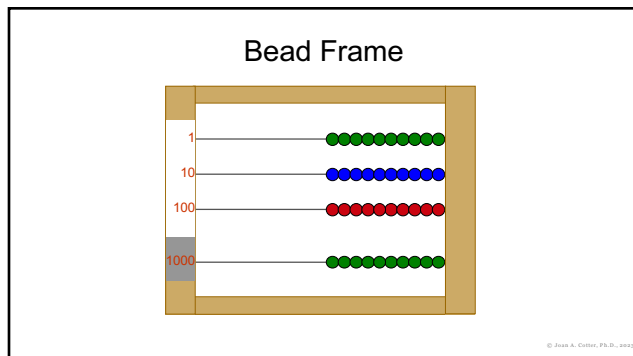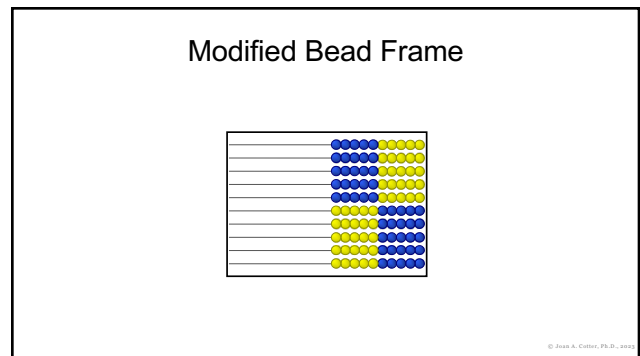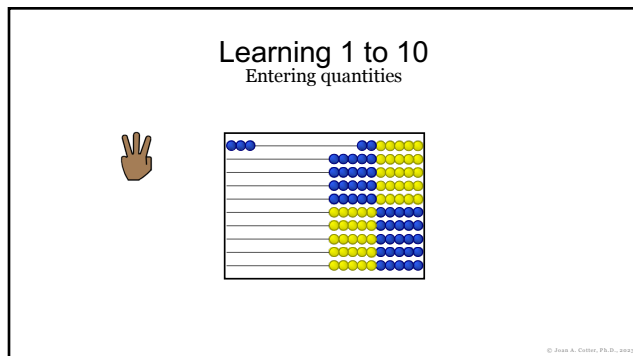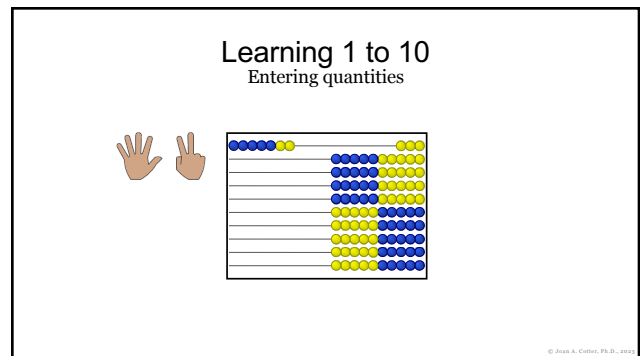
49

Learning 1 to 10
Entering quantities
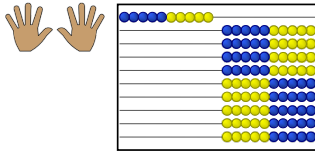
50

Learning 1 to 10
The stairs
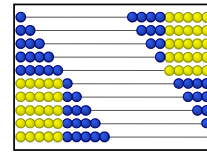
51

Learning 1 to 10
Adding
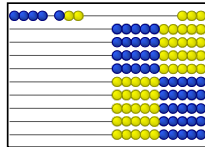
4 + 3 = __

52

Learning 1 to 10
Adding

4 + 3 = _7_
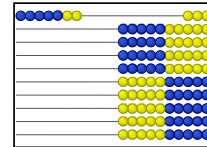
53

Strategies

- A strategy is a way to learn a new fact or recall a forgotten fact.
- Effective strategies are dynamic and visualizable.

54

Addition Strategies

Basic addition strategies:

- Partitioning Ten
- Complete the Ten
- Two Fives

55

Addition Strategies
Partitioning Ten
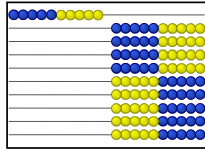
10 = 4 + __

Addition Strategies
Partitioning Ten

10 = 4 + 6

Addition Strategies
Complete the Ten

9 + 5 = __
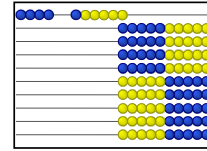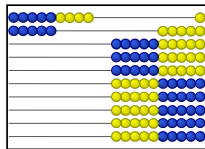
Addition Strategies
Complete the Ten

9 + 5 = __

Take 1 from the 5
and give it to the 9.
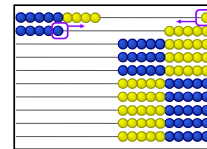
Addition Strategies
Complete the Ten

9 + 5 = 14
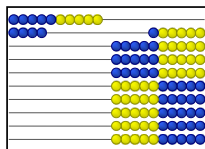
Addition Strategies
Two Fives

8 + 6 =
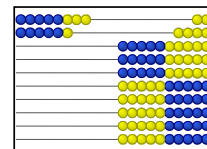
Addition Strategies
Two Fives

8 + 6 =
10 + 4 = 14
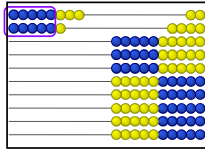
Games

$$\frac{\text{Games}}{\text{Math}} = \frac{\text{Books}}{\text{Reading}}$$

- Games provide instant feedback.
- Games provide interesting repetition needed for automatic responses.
- Games promote learning in a social setting.
- Games provide motivation for learning the facts.

Games

Effective math games:

- Must be fun to play and combine skill and luck.
- Support pupils in learning while they play.
- Include hands-on materials or chart.
- Should not be glorified flash cards.
- Are an ideal place to teach social skills.

Go to the Dump Game

- A "Go Fish" game for practicing facts totaling 10.
- The pairs are: 1 and 9
  2 and 8
  3 and 7
  4 and 6
  5 and 5
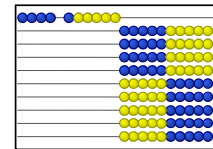
Go to the Dump Game

- Objective: To practice finding sums that total 10.
- Number of Players: Two to five.
- Cards: About 10 to 12 cards of each number 1 to 9.
- Object of the Game: To collect the most pairs totaling 10.
- Layout: Each player takes 5 cards to start.
- Play: Player asks the person on their left for a card they need. If yes, they ask again. If no, they take another card while the other person says 'Go to the Dump' and takes their turn.

Go to the Dump Game

Photo by Cheryl Heatwole Shenk

Go to the Dump Game

68

Five in a Row Game

- Objective: To practice finding sums that total 10.
- Number of Players: A solitaire.
- Cards: Four cards of each number from 1 to 9
- Object of the Game: To pair all the cards.
- Layout: Lay five cards in a row face up. Lay another row of cards directly below the first row but face down.

69

Five in a Row Game
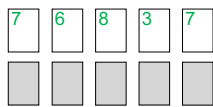
7	6	8	3	7

70

Five in a Row Game
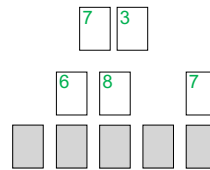
7 3

6 8   7

71

Five in a Row Game
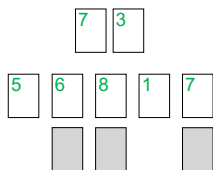
7 3

5 6 8 1 7

72

Five in a Row Game
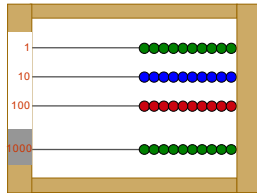
73

Bead Frame

8 + 6 =

74

Bead Frame

8 + 6 =

75

Bead Frame

8 + 6 =

76

Bead Frame

8 + 6 =

77
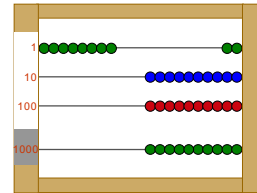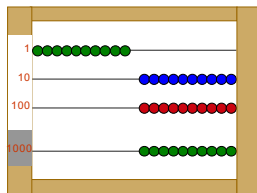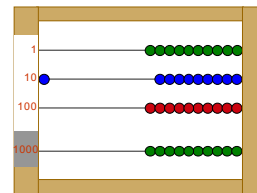
Bead Frame

8 + 6 = 14
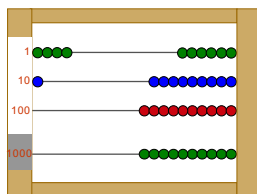
78

Bead Frame

Difficulties for the child
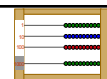
- Not visualizable:
- When beads are moved right, inconsistent with order of the equation:
- Background visible through frame:
- Hierarchies of numbers represented sideways:

- Exchanging done before second number is completely added:
- Answer is read going up:

79

Subtraction Strategies

Basic subtraction strategies:

- Going Up
- Subtracting Part from Ten
- Subtracting All from Ten

80

Subtraction Strategies
Going Up

15 − 9 =
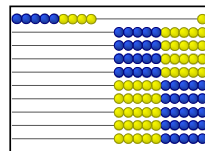
Start with 9;
go up to 15.

81

Subtraction Strategies
Going Up

15 − 9 =
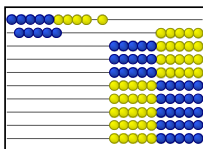
Start with 9;
go up to 15.

82

Subtraction Strategies
Going Up

15 − 9 =
1 + 5 = 6

Start with 9;
go up to 15.
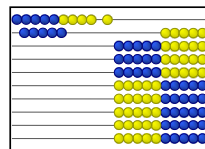
83

Subtraction Strategies
Going Up

11 − 7 =

Start with 7;
go up to 11.
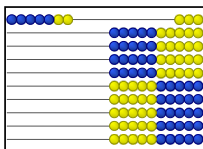
84

Subtraction Strategies
Going Up

11 − 7 =
3 + 1 = 4

Start with 7;
go up to 11.
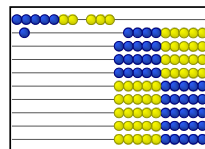
85

Subtraction Strategies
Subtracting Part from Ten

15 − 9 =
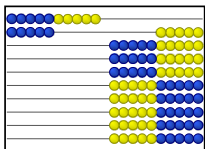
Subtract 5 from 5
and 4 from 10.

86

Subtraction Strategies
Subtracting Part from Ten

15 − 9 =
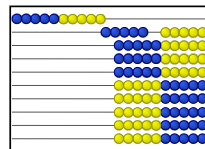
Subtract 5 from 5
and 4 from 10.

87

Subtraction Strategies
Subtracting Part from Ten

15 − 9 = 6

Subtract 5 from 5
and 4 from 10.
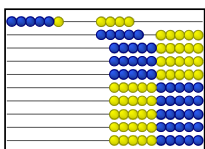
88

Subtraction Strategies
Subtracting Part from Ten

11 − 7 =

Subtract 1 from 1
and 6 from 10.
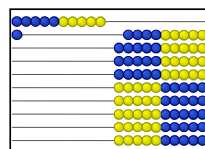
89

Subtraction Strategies
Subtracting Part from Ten

11 − 7 = 4

Subtract 1 from 1
and 6 from 10.
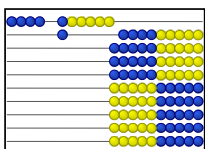
90

Subtraction Strategies
Subtracting All from Ten

15 − 9 =
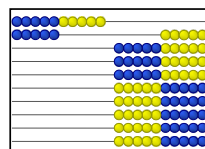
Subtract 9 from 10.

91

Subtraction Strategies
Subtracting All from Ten

15 – 9 =
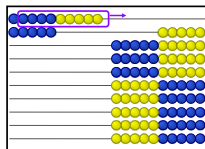
Subtract 9 from 10.

92

Subtraction Strategies
Subtracting All from Ten

15 – 9 = 6

Subtract 9 from 10.
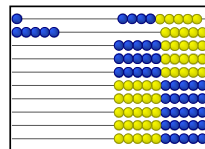
93

Subtraction Strategies
Subtracting All from Ten
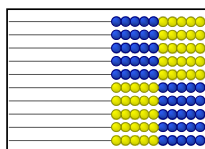
94

Subtraction Strategies
Subtracting All from Ten

11 – 7 =

95

Subtraction Strategies
Subtracting All from Ten

11 – 7 =
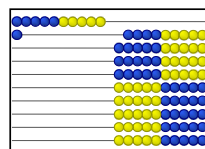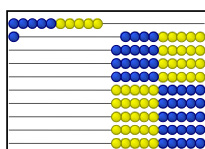
Subtract 7 from 10.

96

Subtraction Strategies
Subtracting All from Ten

11 – 7 =
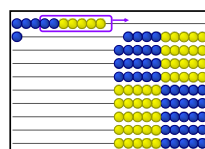
Subtract 7 from 10.

97

Subtraction Strategies
Subtracting All from Ten

11 − 7 = 4

Subtract 7 from 10.
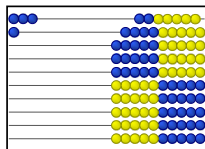
Ten Minus Game

- A "Go Fish" game for practicing subtraction facts: 10 − 1, 10 − 2, 10 − 3, . . . , 10 − 9.
- Play it like "Go to the Dump."
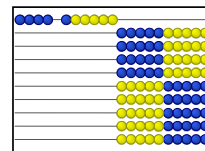
Multiplication Strategies
Visualizing basic facts

6 ✕ 4 = 24
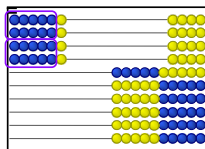
Multiplication Strategies
Visualizing basic facts

9 ✕ 3 = 30 − 3 = 27
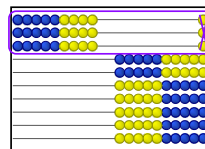
Multiplication Strategies
Visualizing basic facts

7 ✕ 7 = 25 + 10 + 10 + 4 = 49
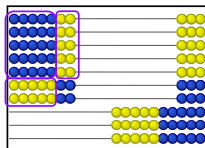
Ring Around the Products Game

- Objective: To practice the multiplication facts.
- Number of Players: Two to four.
- Cards: Multiplication cards 1, 2, . . . , 90, 100 and about 10 to 12 cards of each number 1 to 10.
- Reference: A multiplication chart.
- Object of the Game: To collect the most multiplication cards.

Ring Around the Products Game

104

Ring Around the Products Game

105

Ring Around the Products Game

106

The Multiplication Board

6 × 4

Not visualizable.

107

The Multiplication Board

6 × 4
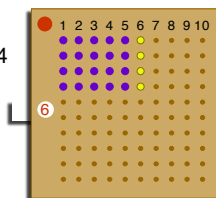
Now is visualizable.

108

The Multiplication Board

7 × 7
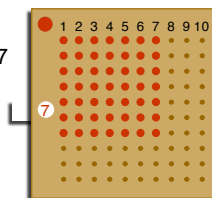
Not visualizable.

109

The Multiplication Board

7 × 7

1 2 3 4 5 6 7 8 9 10

⑦

Is visualizable.

© Juan A. Cotter, Ph.D., 2023

110

Multiplication Chart

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

8 × 8 = 64

© Juan A. Cotter, Ph.D., 2023

111

Multiplication Chart

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

8 × 8 = 64

© Juan A. Cotter, Ph.D., 2023

112

Summary

- Infants naturally subitize and visualize; we must help children nurture these gifts.

- Subitizing and visualizing reduce memory load and greatly help our disadvantaged children.

- Visualizing requires grouping in fives.

- Effective strategies for mastering the facts are visualizable.

- Visualizing is critical for attaining abstraction.

© Juan A. Cotter, Ph.D., 2023

113

Thank you!

Contact Information

Joan A. Cotter, Ph.D.

✉ JoanCotter@RightStartMath.com

📞 701-321-1406

🌐 www.RightStartMath.com

https://tinyurl.com/ywax7kju

Cheryl Heatwole Shenk, M.Ed

✉ CHeatwoleShenk@CSPSchools.com

📞

🌐 www.MontessoriSprouts.com