New Ways of Teaching and Learning Basic Arithmetic Using Visualizable Strategies

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Abstract

Confusion with number sense is often due to a vague understanding of what numbers mean and how they relate to each other. Attempts have been made to solve this by focusing on rote memorization. Yet, children who have memorized without understanding struggle to apply their skills. So often the student is taught to recognize the numeral 7 and the numeral 6 then memorize the answer as a 1 and a 3. Later in the year, the numeral 7 and the numeral 6 has a memorized answer of 1, and a year later, an answer of a 4 and a 2. This often results in frustration, confusion, and an aversion to math.

There is a solution for success in learning the facts: subitizing, visualizing, and using strategies. Subitizing is the base on which to build; then, strategies help organize the numbers. Visualization gives the results in a format that can be easily recalled.

Introduction

So many students (and adults) have a weak understanding of basic arithmetic. They are slow and sometimes inaccurate in recalling the facts. They often have low comprehension and struggle to apply arithmetic to daily living and in higher level mathematics.

To overcome this hurdle, students are expected to rote memorize the 390 math facts. Sadly, whatever is learned by rote needs frequent review. This required memorization decreases the chance to experience the joy of mathematics.

So often, counting is thought to be the precursor to memorization. To experience adding with counting from a child's perspective, use the letters of the alphabet rather than numbers. So, A is 1, B is 2, C is 3, and so forth. Use counters to find F + E. Proceed by counting out F counters, then E counters, and counting them all to obtain the sum K as shown in Figure 1 (Cotter, 2022).

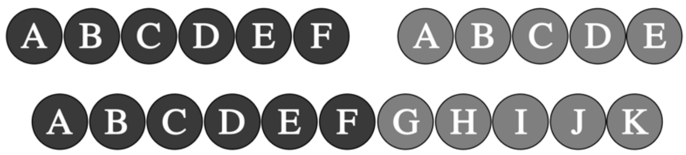


Figure 1. F + E = K

Now try to add C + D without counters; however, finger counting is allowed. Then try G + G without fingers.

Subtraction is no easier. Try solving J – F by counting back. First J counters need to be counted out. Then count back: I, H, G, F, E, D. This leaves D as the difference.

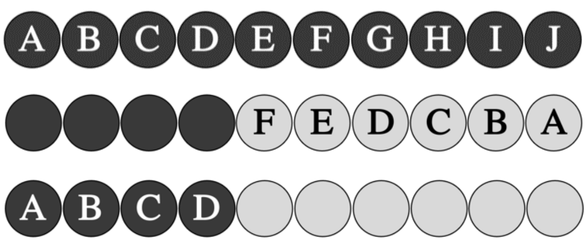


Figure 2. J – F = D

Another option is counting out J counters, then F counters need to be removed, leaving an amount to count to find the difference of D. See Figure 2.

No wonder why our students are overwhelmed! Multiplication using this approach is nearly impossible.

Then, flash cards are produced. Do you know the answers? See Figure 3. Are you seeing the beauty of math here? Or are you overwhelmed and experiencing anxiety? People have reported their math anxiety started with flash cards and timed tests. Sadly, many adults live with some level of math anxiety.

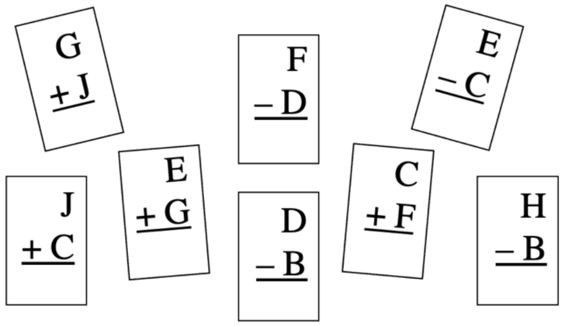


Figure 3. Sample flash cards

Subitizing, Visualizing, and Strategies

Subitizing, the rapid and confident recognition of a quantity without counting, was identified by E. L. Kaufman et al. in 1949. When quantities are grouped by fives, amounts greater than five can also be subitized. Subitizing beyond ten is also done by grouping. Humans, especially children, think visually. Our human brains need grouping to “see” quantities. What can be subitized can be visualized.

Visualizing is the ability to form a mental image, to imagine. An important part of math is visualizing – seeing in your mind. It is well known that most of us learn best with visualizable images. Visualizing is also important for critical thinking.

A strategy is a way to learn a new fact or recall a forgotten fact. A visual representation is a powerful strategy. Subitizing supports visualization; visualization and strategies are interdependent.

Partitioning on the Cotter Abacus

The Cotter Abacus is grouped by fives and tens to promote subitizing. See Figure 4. Remember, what can be subitized can be visualized. This makes the Cotter Abacus visualizable.

Ten can be partitioned on the abacus, which quickly translates to strategies for the facts of ten: 1 + 9, 2 + 8, 3 + 7, and so on. See Figure 5.

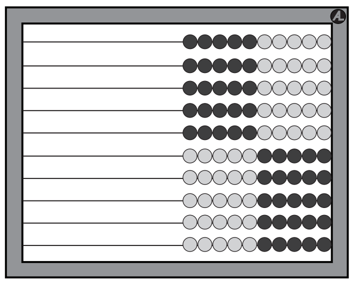


Figure 4. Cotter Abacus

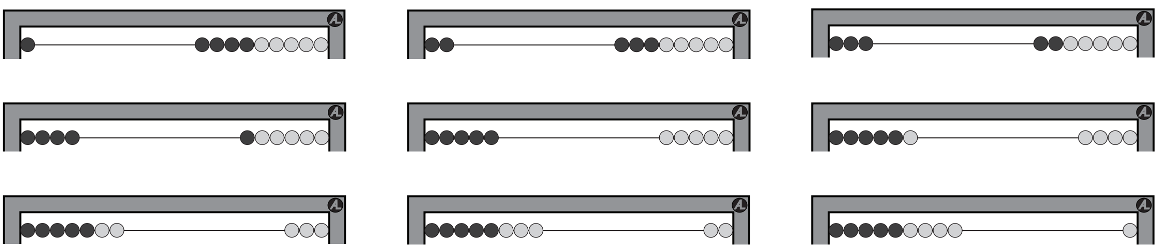


Figure 5. Ten partitioned, supporting the facts of ten

One hundred can also be partitioned on the Cotter Abacus. See Figure 6. The left image shows 68 + 32, which equals 100 and the second image shows 45 + 55. Notice how the grouping in fives and tens allows the two quantities to be quickly subitized. It also shows the relationship of the quantities of 45 and 55 with the whole of 100.

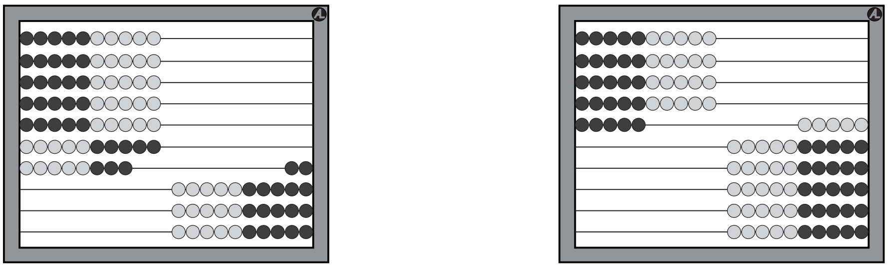


Figure 6. 68 + 32 = 100 and 45 + 55 = 100

Mastering Addition Facts

Adding numbers with sums of 10 or less is simply a matter of entering the beads and subitizing the total. Figure 7 shows 5 + 2 and 3 + 4, both quickly recognized as a sum of 7.

Figure 7. 5 + 2 = 7 and 3 + 4 = 7

For sums over ten, there are two strategies which incorporate subitizing. The first strategy is Complete the Ten. This strategy's goal is to take from the smaller number to make the larger number into a 10. Consider 9 + 5 as shown in the first image in Figure 8. Take 1 from 5 and give it to 9 which makes 10 + 4 = 14.

The second strategy for sums over ten is the Two Fives strategy. In the second image of Figure 8, see 8 + 6. Consider 8 as a group of 5 and 3 more and 6 as a group of 5 and 1. Combine the two 5s to make 10 and the leftover 3 and 1 to make 4, giving the sum of 10 + 4, or 14.

Figure 8. Complete the Ten strategy and Two Fives strategy

These strategies can be applied to higher numbers. See figure 9. The first image shows 54 + 9 and uses the Complete the Ten strategy to find the sum of 63. The second image shows 35 + 7 and uses the Two Fives strategy. Four tens are present, three full tens and a fourth ten from the two 5s, and 2 more for a sum of 42.

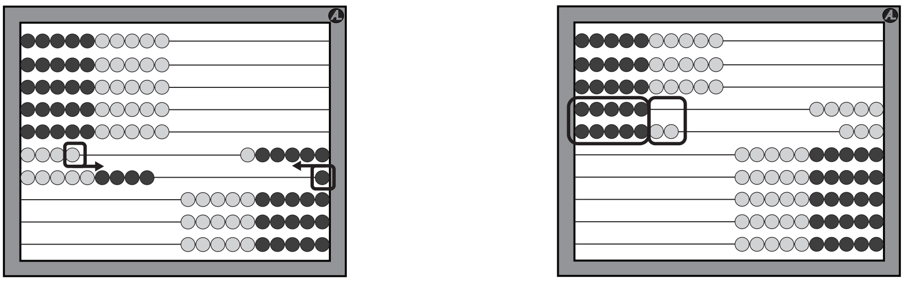


Figure 9. Complete the Ten strategy and Two Fives strategy with higher decades

Keep in mind that a student is considered to know a fact if the answer is given in two to three seconds. This means that fact retrieval does not need to be instantaneous. This two to three second thinking time is enough for the student to think through a strategy when necessary.

Mastering Subtraction Facts

Subtraction can be thought of as finding a part when the whole and the other part are known. Three subtraction strategies are available for mastering the subtraction facts.

The first subtraction strategy is Going Up. Consider 15 – 8 as the example. See the first image in Figure 10. Start with 8, go to 10 and note that 2 is needed to reach 10, and then 5 more is needed to go from 10 to 15. Combining 2 + 5 is 7, so 15 – 8 = 7.

The second strategy is Taking Part from Ten. Using the same example, 15 – 8, subtract 5 from 15, leaving 10. Subtract the remaining 3 from the 10, leaving 7 as the difference. See the second image in Figure 10.

The third strategy is Taking All from Ten. Again, using the same example, 15 – 8, subtract the entire 8 from 10, leaving 2, and add that to the remaining 5, again getting 7. See the last image in Figure 10.

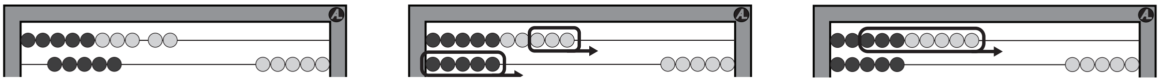


Figure 10. Going Up, Taking Part from Ten, and Taking All from Ten strategy

The All from Ten strategy was used extensively in the Middle Ages. Back then, they did not memorize any facts subtracting from 11 to 18; they always used the Taking All from Ten strategy.

Mastering Multiplication Facts

Multiplication has been the mathematical downfall of many students (and adults). It is not so much the problem of the algorithms, but rather it is the problem of memorizing the 100 facts.

Before expecting the student to learn the facts, we need to teach the meaning of multiplication. So often, we say it is the same as repeated addition. Yet, describing multiplication as repeated addition provides a limited view. An array or an arrangement of objects in rows and columns makes a better model. A row with six objects repeated three times is 6 multiplied by 3, or six taken three times. This array produces a product of 18.

There are different interpretations about the meaning of 6 × 3. Sometimes 6 × 3 is thought of as 6 groups of 3, rather than 6 repeated 3 times. Let's compare the meaning of multiplication to that of other arithmetic operations:

When we add 6 + 3, we start with 6 and transform it by adding 3 to it.
When we subtract 6 – 3, we start with 6 and transform it by decreasing 3.
When we divide 6 ÷ 3, we start with 6 and transform it by dividing it into either 3 groups or groups of 3.

Therefore, to be consistent, when we multiply 6 × 3, we start with 6 and transform it by duplicating it 3 times. This interpretation also corresponds to the

Cartesian coordinate system. Representing the 6 × 3 array with 6 across in 3 rows is similar to finding a point (6, 3) on a grid. The first number, 6, indicates the horizontal number and the 3, the vertical number.

For learning the multiplication facts, the commutative property simplifies the task. The commutative property reduces the number of facts to be learned in a 10 by 10 multiplication table from 100 facts to 55 facts.

Learning the 1s facts is easy. 1 × 8 means 1 repeated 8 times. Answer is 8. 8 × 1 means 8 taken 1 time, which also is 8.

The 2s facts are already known from the addition facts. 4 × 2 is the same as 4 + 4 and 9 × 2 is the same as 9 + 9. The commutative property extends to 2 × 4 = 8 and 2 × 9 = 18.

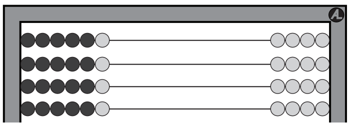
The 10s facts are known from place-value work. 10 × 3 can be viewed as 10 three times, or 30. This leaves only 28 facts to learn.

The 4s facts are the 2s facts doubled. 4 × 3 can be thought of as 2 × 3, which is 6, and doubling that for a total of 12. 4 × 7 is 2 × 7, 14, then doubled is 28.

The 8s facts are the 4s facts doubled. This means that 8 × 3 is 4 × 3, which is 12, then doubled for a product of 24 and 8 × 6 is the same as 4 × 6 doubled.

The Cotter Abacus provides solid visualizable strategies. For example, enter 6 × 4 on the abacus. Remember, this means 6 duplicated 4 times. See the first image in Figure 11.

Adults generally think in pictures, and children definitely think in pictures. This picture of 6 × 4 gives the child the image of 6 × 4 and a way to find the product. Notice the two groups of 10s and the remaining four ones. This makes 6 × 4 equal to 10 × 2 plus 4 more, which is 24.



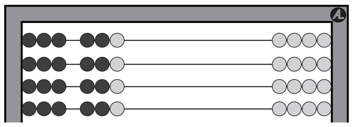


Figure 11. 6 × 4 = 20 + 4 = 24 and 6 × 4 = 3 × 4 doubled, which is 24

Another approach to 6 × 4 is thinking of 3 × 4, which is 12, and doubling that for a total of 24. See the second image in Figure 11.

The fact 9 × 4 can be seen as 10 × 4, which is 40, less 4 which are not included to give 36. See Figure 12.

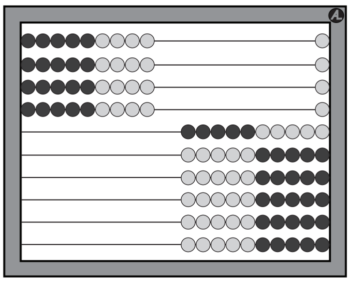


Figure 12. 9 × 4 = 40 – 4 = 36

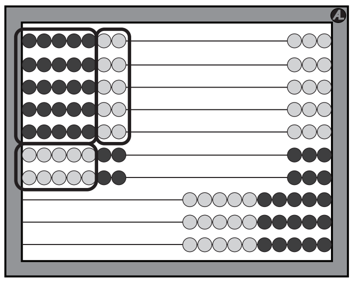


Figure 13. 7 × 7 = 25 + 10 + 10 + 4 = 49

One last example is 7 × 7. See Figure 13 on the previous page. Here you can see the five rows of five dark beads that make 25. Also, see 10 light beads along the right two columns and another 10 in the bottom two rows. Finally, the lower right corner has 4 more. It takes less than two or three seconds to summon the visual image of 7 × 7 and find the product: 25 + 10 + 10 + 4 = 49.

Remember, a child is considered to know a fact when they can answer in two or three seconds.

Summary

Grouping in 5s and 10s is the foundation for subitizing and subitizing is the foundation for visualizing. A strategy is a procedure to learn a fact or to recall a forgotten fact. A strategy is made more powerful when a visual representation is incorporated. Visualization gives the answers in a format that can be easily recalled.

We need to teach the basic arithmetic facts so that the children develop a reliance on subitizing, strategies, and visualization. These three aspects will give our students a new way of learning.

Ignacio Estrada from Gordon and Betty Moore Foundation says, “If a child can’t learn the way we teach, maybe we should teach the way they learn.”

Joan A. Cotter, curriculum developer and author, says, “Our goal as a teacher of mathematics is to help our children transform, expand, and refine these beginning ideas into deeper mathematical thinking.”

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