A Deep Dive into a Critical Component of Arithmetic: Place Value

Joan A. Cotter, Ph.D.
JoanCotter@RightStartMath.com

River Falls, Wisconsin
January 18, 2025
2:00 – 3:15 p.m.

My Background

- AMI credentialed in early childhood
- Ph.D. in mathematics education
- MACI in curriculum and instruction
- BSEE in electrical engineering
- Mathematics teacher for grades 6–8
- Author of K–8 RightStart Mathematics program

2

Rationale

"This book of methods compiled by one person alone, must be followed by many others. It is my hope that, . . . , other educators will set forth the results of their experiments. These are the pedagogical books which await us in the future."

– Maria Montessori

The Montessori Method: Scientific Pedagogy as Applied to Child Education in "The Children's Houses," 1912.

3

Rationale

- Down through history all great movements have struggled to stay relevant.
- The task is to keep their core values while embracing new knowledge and engaging the next generation.
- Note that the U.S. Constitution provides for changes through amendments.

4

The Basics of Place Value

- Place value, not counting, is the key to understanding numbers beyond ten.
- Without place value we are giving children numbers in bulk, rather than tidy packages.
- Children need to learn *place* value, not *color*, value.
  - One out of 12 boys has some color deficiency.
  - One out of 200 girls has some color deficiency.

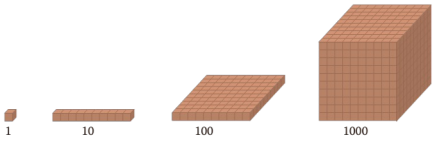
5

The Basics of Place Value

- The author of *Treviso Arithmetic of 1478* considered place value so important that it was listed first among the "five" operations of arithmetic.
- Place value organizes numbers into neat packets.
- It is a necessary ingredient to make sense of computational algorithms.
- Children need to appreciate how special 10 is.
- Place value needs to be incorporated in learning the facts.

6

The Basics of Place Value



The number of zeros in each representation matches the number of dimensions.

- 1 has 0 zeros and 0 dimensions.
- 10 has 1 zero and 1 dimension (line).
- 100 has 2 zeros and 2 dimensions (area).
- 1000 has 3 zeros and 3 dimensions (volume).

7

The Basics of Place Value

The dandelion plant approximates the 3 dimensions.


Seed → point:
0 dimensions


Stem → line:
1 dimension


Flower → circle:
2 dimensions


Puffball → sphere:
3 dimensions

8

The Basics of Place Value

321,987,654,321

Billions			Millions			Thousands			Ones		
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones
3	2	1	9	8	7	6	5	4	3	2	1

9

The Basics of Place Value

Patterning

- Continue the pattern:
  - 1 1
    - 1 1 1 1
    - 1 1 2 3 5 8 13
  - 2 4
    - 2 4 6 8
    - 2 4 8 16 32 64
- We need more than 2 elements to continue a pattern. That's why we need thousands to understand place value, which the Montessori materials do!

10

The Basics of Place Value

- Static (Recording)
  - Value of a digit is determined by position.
  - No position may have more than nine.
  - As you progress to the left, value at each position is 10 times greater than the previous position.
- Dynamic (Trading)
  - 10 ones = 1 ten; 10 tens = 1 hundred; 10 hundreds = 1 thousand, ….
  - During computation, a position may have more than 9.

11

The Basics of Place Value

Common Core state standards:

- First grade: "Understand the following as special cases: 10 can be thought of as a bundle of ten ones — called a 'ten.'"
- Second grade: "Understand the following as special cases: 100 can be thought of as a bundle of ten tens— called a 'hundred.'"
- Fourth grade: "Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right."

12

Research

- Montessori children showed an advantage on correct use of base-10 representation in kindergarten but not in first grade.
  - Children were asked to construct 12, 16, 28, 34, and 61 with 10s and 1s.
- At the end of E1: ". . . no difference remained in place-value understanding between Montessori and non-Montessori children in third grade."

Longitudinal Comparison of Place-Value and Arithmetic Knowledge in Montessori and Non-Montessori Students *Journal of Montessori Research.* Laski, Vasilyeva, and Schiffman. 2016.

13

Research

- Researchers suggest give "clear explanations of the relation between the manipulative and the concept it represents."
- Avoid unnecessary repetition: "transition to abstract representations more rapidly."
- Explain the "use of decomposition for mentally solving addition problems."

Longitudinal Comparison of Place-Value and Arithmetic Knowledge in Montessori and Non-Montessori Students *Journal of Montessori Research.* Laski, Vasilyeva, and Schiffman. 2016.

14

Making 10 More Prominent

To appreciate place value, children must appreciate the specialness of 10.

- Subitizing rather than counting.
- Making 10 visible through grouping in 5s.
- Use transparent number naming temporarily for numbers greater than 10.
- Teach 10-based strategies for mastering the facts.

15

Subitizing

- Karen Wynn found that infants can determine quantities long before they can count or even talk.
- They can add and subtract 1 to 3 objects without counting.
- This ability to recognize quantities without counting is called *subitizing*.
- Subitizing quantities makes them visualizable.

16

Grouping in Fives

Without counting or grouping, name the number of apples.

17

Grouping in Fives

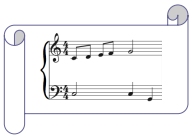
Now name the number of apples.

18

Grouping in Fives

Early Roman numerals

1	I
2	II
3	III
4	IIII
5	V
8	VIII



19

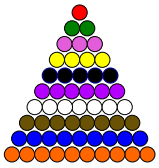
Grouping in Fives

"Grouped in fives so the child does not need to count."

– A. M. Joosten
Director of Indian Montessori and Montessori Training Center of Minnesota (now Montessori Center of Minnesota)

20

Grouping in Fives



- Associating quantities with colors is not subitizing– it obscures visual understanding of quantity.
- Using colors for quantities is NOT generalizable.

21

Grouping in Fives

Chunking:

- Grouping in fives provides "chunking."
- According to psychologists, a chunked item can be stored or processed as one item.
- Ten can be reduced to two chunks, reducing memory load.

22

Grouping in Fives

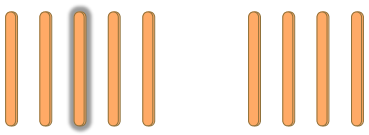
In Japanese schools:

- Students consistently group in fives.
- Children are discouraged from using counting for adding.
- They are not taught to count on.

23

Grouping in Fives

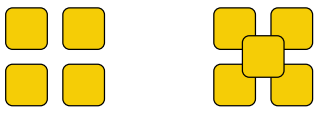
Distinguishing 5 from 4:



5 has a middle; 4 does not.

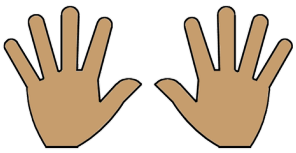
24

Grouping in Fives



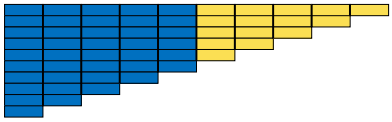
25

Grouping in Fives



26

Grouping in Fives



27

Grouping in Fives

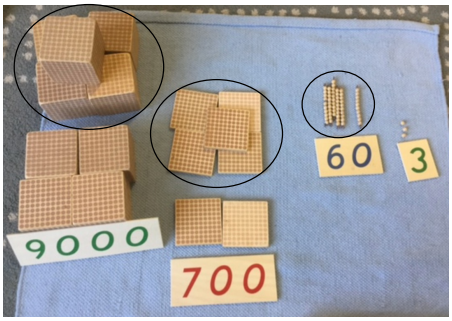
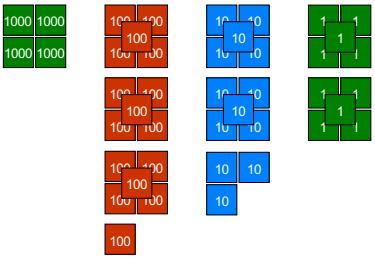


Photo by Cheryl Heatwole Shenk

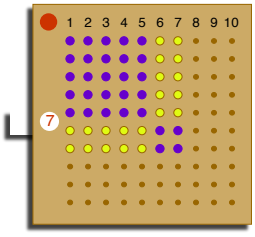
28

Grouping in Fives



29

Grouping in Fives



7 □ 7 = 25 + 10 + 10 + 4 = 49

30

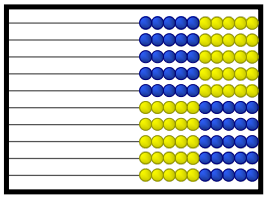
Grouping in Fives

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

8 □ 7 =

31

Grouping in Fives



32

Learning 1 to 10

33

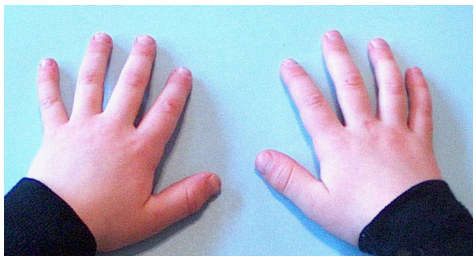
Learning 1 to 10

34

Learning 1 to 10

35

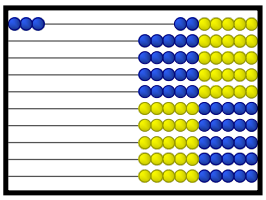
Learning 1 to 10



36

Learning 1 to 10

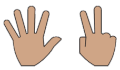
Entering quantities

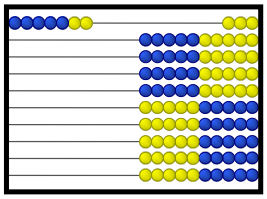


37

Learning 1 to 10

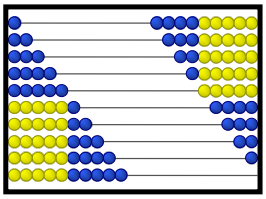
Entering quantities





38

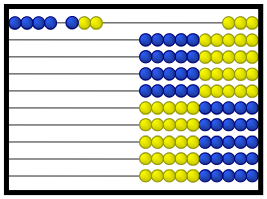
Learning 1 to 10



39

Learning 1 to 10

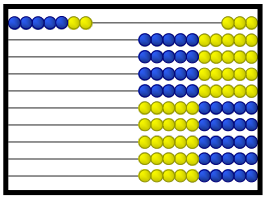
4 + 3 = ___



40

Learning 1 to 10

4 + 3 = 7



41

The Counting Model

Counting is pervasive in Montessori materials:

- Number Rods
- Spindle Boxes
- Golden Bead materials
- Snake Game
- Dot Game
- Stamp Game
- Multiplication Board
- Bead Frame

42

The Counting Model

Counting:

- Is not natural; it takes years of practice.
- Provides poor concept of quantity.
- Ignores place value.
- Is very error prone.
- Is tedious and time-consuming.
- Does not provide efficient ways to master facts.
- Is not predictive of future math achievement.

43

The Counting Model

Counting on:

- Children are often expected to know what number comes next without starting from one.
- Think of the nursery rhyme "Jack and Jill."
- Without starting from the beginning, what word comes after hill?

44

The Counting Model

Counting is especially difficult for children:

- With dyslexia or dyscalculia
- With SLI (specific language impairment)
- With memory challenges
- With poor motor control
- From low SES backgrounds

45

The Counting Model

Is counting the core of math?

- Counting is used primarily in arithmetic.
- Arithmetic is one of about 200 branches of math.
- Abacuses made counting unnecessary.
- It is very slow for multiplying.
- Counting doesn't work for money or fractions.

46

Transparent Number Naming

10 = ten
11 = ten 1 or 1-ten 1
12 = ten 2 or 1-ten 2
13 = ten 3 or 1-ten 1
14 = ten 4 or 1-ten 1
. . . .
19 = ten 9 or 1-ten 1
20 = 2-ten
21 = 2-ten 1
22 = 2-ten 2
23 = 2-ten 3
24 = 2-ten 4
. . . .
30 = 3-ten
31 = 3-ten 1
32 = 3-ten 2
33 = 3-ten 3
. . . .
99 = 9-ten 9

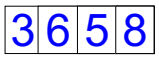
47

Transparent Number Naming

- Children in East Asia learn mathematics using transparent number naming.
- They understand place value in first grade; often before they start school.
- Mathematics is the science of patterns. The pattern in transparent number naming greatly helps children learn number sense.

48

Transparent Number Naming



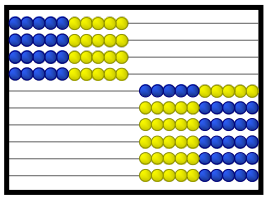
- We need to be sure children learn *place* value, not *color* value.
- One out of 12 boys has some color deficiency.
- So do one out of 200 girls.

49

Transparent Number Naming

4-ten = forty

The "ty" means groups of ten.

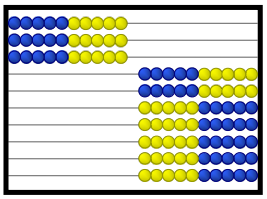


50

Transparent Number Naming

3-ten = thirty

"Thir" also used in 1/3 and 13.

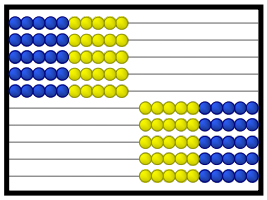


51

Transparent Number Naming

5-ten = fifty

"Fif" also used in 1/5 and 15.

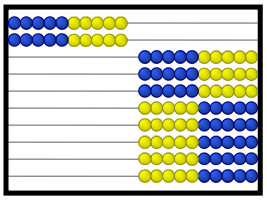


52

Transparent Number Naming

2-ten = twenty

"Two" used to be pronounced "twoo."



53

Transparent Number Naming

A word game:

fireplace → place-fire

newspaper → paper-news

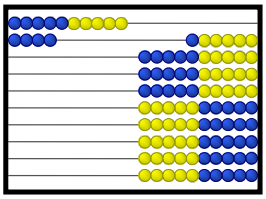
box-mail → mailbox

54

Transparent Number Naming

ten 4 → teen 4 → fourteen

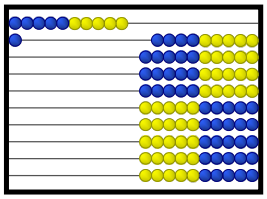
"Teen" also means ten.



55

Transparent Number Naming

a one left ⟶ a left-one ⟶ eleven

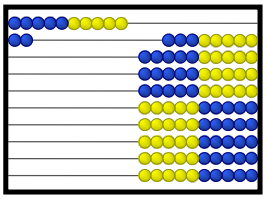


56

Transparent Number Naming

two left ⟶ twelve

"Two" said as "twoo."



57

Can You Find Game

Can you find 6 hundred?
2 hundred 5?
3-ten 9?
8 hundred 46?

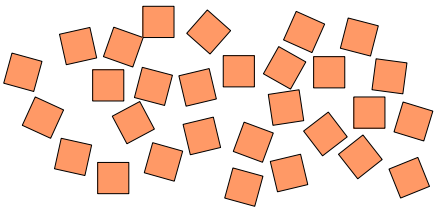
600
205
39
846

3000
80
500
40
2
2000
90
20
60
7
400
800
700
1
1000
8
100
3
70
300
50
900
6
10
4

58

Applying Place Value

A typical problem in Japanese schools

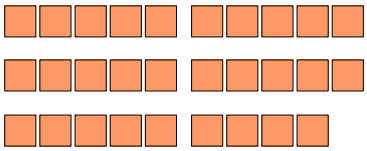


"Can you arrange these tiles so I can tell how many there are without counting?"

59

Applying Place Value

A typical problem in Japanese schools

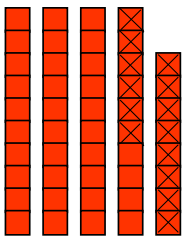


"Can you arrange these tiles so I can tell how many there are without counting?"

60

Applying Place Value

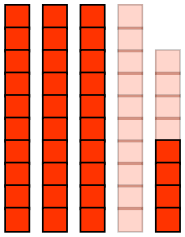
- Using 10s and 1s, ask the child to construct 48.
- Then ask the child to subtract 14.
- A child thinking of 14 as 14 ones will count 14.



61

Applying Place Value

- Using 10s and 1s, ask the child to construct 48.
- Then ask the child to subtract 14.
- A child thinking of 14 as 14 ones will count 14.
- A child using place value will remove ten and 4 ones.



62

Mastering the Facts

- Rote memorizing needs lots of review.
- Mnemonics require an extra step.
- Visualizable strategies are always available.
- Children have 2-3 seconds to retrieve a fact.
- Good strategies employ place value.
- Practice can be done through games.

63

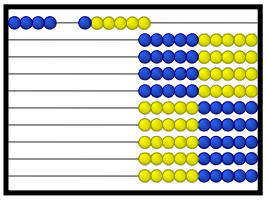
Mastering the Facts
Strategies

- A strategy is a way to learn a new fact or recall a forgotten fact.
- Effective strategies are dynamic and visualizable.
- Good strategies provide an opportunity to apply place value.

64

Addition Strategies
Partitioning Ten

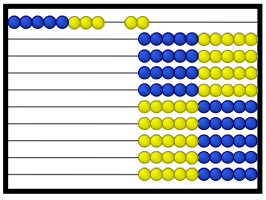
10 = 4 + 6



65

Addition Strategies
Partitioning Ten

10 = 8 + 2

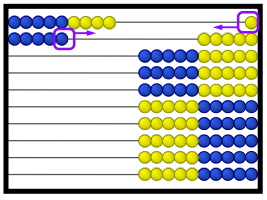


66

Addition Strategies
Complete the Ten

9 + 5 = __

Take 1 from the 5 and give it to the 9.

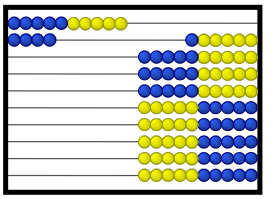


67

Addition Strategies

Complete the Ten

9 + 5 = 14

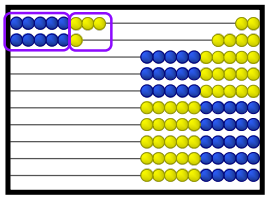


68

Addition Strategies

Two Fives

8 + 6 =
10 + 4 = 14



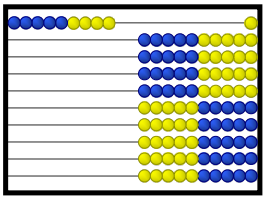
69

Subtraction Strategies

Going Up

15 – 9 =

Start with 9;
go up to 15.



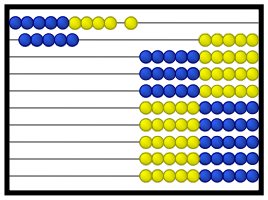
70

Subtraction Strategies

Going Up

15 – 9 =
1 + 5 = 6

Start with 9;
go up to 15.



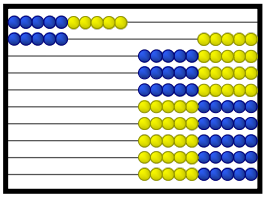
71

Subtraction Strategies

Subtracting Part from Ten

15 – 9 = __

Subtract 5 from 5
and 4 from 10.



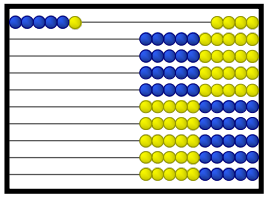
72

Subtraction Strategies

Subtracting Part from Ten

15 – 9 = 6

Subtract 5 from 5
and 4 from 10.



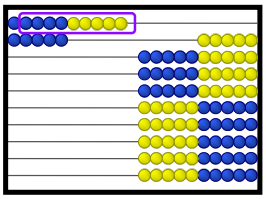
73

Subtraction Strategies

Subtracting All from 10

15 – 9 =__

Subtract 9 from 10.



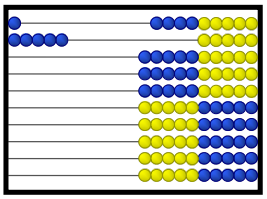
74

Subtraction Strategies

Subtracting All from 10

15 – 9 = 6

Subtract 9 from 10.

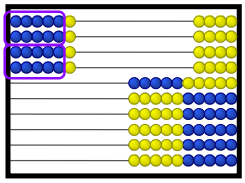


This method was used in the Middle Ages,
rather than memorize the subtraction facts > 10.

75

Multiplication Strategies

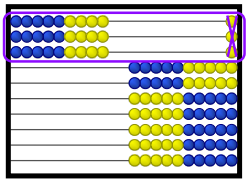
6 × 4 = 20 + 4 = 24



76

Multiplication Strategies

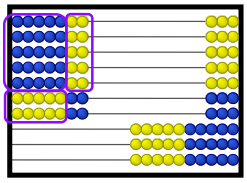
9 × 3 = 30 – 3 = 27



77

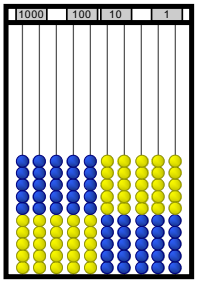
Multiplication Strategies

7 × 7 = 25 + 10 + 10 + 4 = 49



78

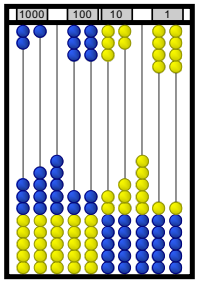
Modified Bead Frame, Side 2



Cleared

79

Adding 4-Digit Numbers

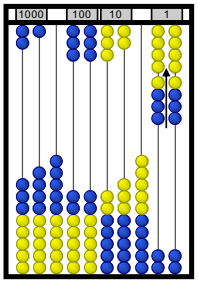


3658
+ 2738

Enter the first number from left to right.

80

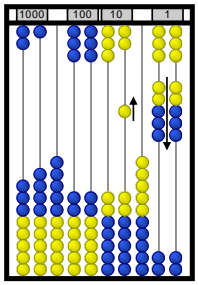
Adding 4-Digit Numbers



3658
+ 2738

81

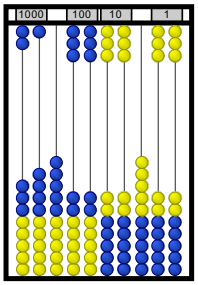
Adding 4-Digit Numbers



3658
+ 2738

82

Adding 4-Digit Numbers

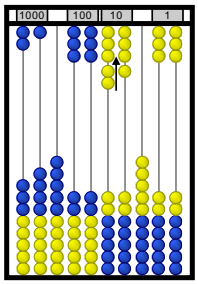


1
3658
+ 2738
6

Write results after each step.

83

Adding 4-Digit Numbers

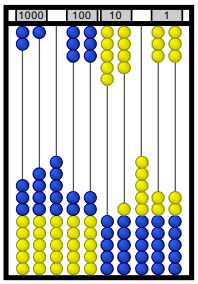


1
3658
+ 2738
6

Write results after each step.

84

Adding 4-Digit Numbers

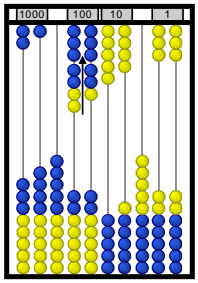


1
3658
+ 2738
96

Write results after each step.

85

Adding 4-Digit Numbers

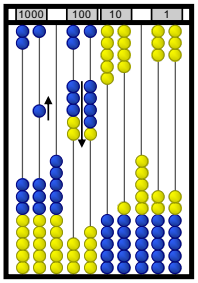


1
3658
+ 2738
96

Write results after each step.

86

Adding 4-Digit Numbers

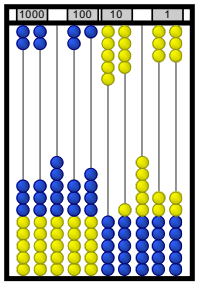


1
3658
+ 2738
96

Write results after each step.

87

Adding 4-Digit Numbers

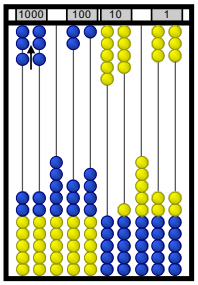


1 1
3658
+ 2738
396

Write results after each step.

88

Adding 4-Digit Numbers

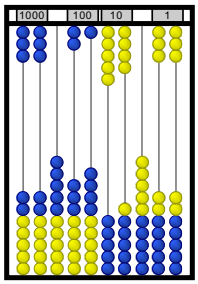


1 1
3658
+ 2738
396

Write results after each step.

89

Adding 4-Digit Numbers



1 1
3658
+ 2738
6396

Write results after each step.

90

Mental Adding

- When numbers are given horizontally:
- Example 1: 42 + 36
  - Add the tens first. 40 + 30 = 70
  - If trading is needed, add 10.
  - Add the ones together, ignoring any 10s. 70 + 2 + 6 = 78
- Example 2: 38 + 24
  - Add the tens first. 30 + 20 = 50
  - If trading is needed, add 10. 50 + 10 = 60
  - Add the ones together, ignoring any 10s. 60 + (1)2 = 62

91

Mental Adding

- When numbers are given verbally:
- Example 1: 42 + 36
  - Add the tens of the second number to the first number. 42 + 30 = 72
  - Then add the ones of the second number. 72 + 6 = 78
- Example 2: 38 + 24
  - Add the tens of the second number to the first number. 38 + 20 = 58
  - Then add the ones of the second number. 58 + 4 = 62

92

Summary

- Subitizing reduces memory load and greatly helps our disadvantaged children.
- Subitizing is needed for visualizing, which is critical for attaining abstraction.
- Effective strategies for mastering the facts use place value.
- We must find more places to use place value.

93

Conclusion

- Dr. Montessori made invaluable contributions to the welfare of children.
- Every great movement struggles with staying true to core principles while staying relevant.
- The content and pedagogy of science and mathematics must be updated periodically.
- We need to use subitizing, grouping, and place value wherever possible.

94